Lecture 1

Monday, 26 January 2026   8:41 AM

Admin Stuff

Classes Tu & Thu 11:30 – 1

Ok for qualifiers

Assessment: 4-5 HWs, 1 in-class exam, Seminar
(50%) (25%) (25%)

Can work together on HWs, but:
- don't search online for solutions
- write yourself (no copying)
- cite references (if you discussed w/ sbdy, or looked up a theorem)

This course:

Linear programming techniques
- possibly the most impt tool in algorithms, OR, combinatorial optimization
- for many optimization problems, first approach should be to write it as an LP.
- looking at the dual LP is often useful

6-7 lectures on LPs
- modelling, geometry, solving LPs, duality, properties incl. total unimodularity.

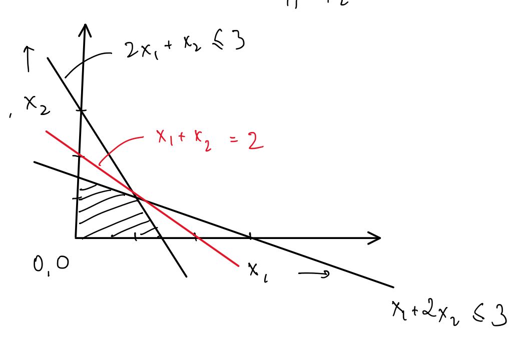
Will then move on to algos using LPs.

Examples, graphical solutions

Geometry is v. impt., should be able to see & visualize the problem

eg. maximize $x_1 + x_2$

subject to
$$x_1 + 2x_2 \leq 3$$
$$2x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0$$



fix c, consider the line $x_1 + x_2 = c$ (slope -1, $\perp$ to $(1,1)$)
max c s.t. a feasible x_1, x_2 is on this line & note that this occurs on a "corner" of the feasible region

HW: Graphically solve the LP

max $x_1 + x_2$

s.t.
$$x_1 + 2x_2 \leq 3$$
$$3x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Forms of LPs

General form: using vector notation, all vectors are column vectors

$$\min_{x \in \mathbb{R}^n} c^T x$$

s.t.
$$a_i^T x \leq b_i \quad \forall i \in M_1$$
$$a_i^T x = b_i \quad \forall i \in M_2$$
$$x_j \geq 0 \quad \forall j \in N_1$$
$$x_j \leq 0 \quad \forall j \in N_2$$

where $M_1, M_2 \subseteq \mathbb{Z}_{++}$ are some index sets

$N_1, N_2 \subseteq [n]$ index sets over the variables

given $c, (a_i)_{i \in M_1 \cup M_2}, N_1, N_2$

note:
$$\max c^T x \equiv \min -c^T x$$
$$a^T x = b \equiv a^T x \geq b \text{ AND } a^T x \leq b$$
$$a^T x \geq b \equiv -a^T x \leq -b$$

defns: objective; constraints; feasible region; free variables

Standard form:
$$\min c^T x$$
$$a_i^T x = b_i \quad \forall i \in [m]$$
$$x_j \geq 0 \quad \forall j \in [n]$$

thus: all constraints are equalities
all variables are nonnegative

Can convert any LP to an LP in standard form, possibly by adding variables and constraints

① Removing free variables:
x_j free, replace x_j by $x_j^+ - x_j^-$
& add constraints $x_j^+ \geq 0$, $x_j^- \geq 0$

② Removing inequalities
convert all inequalities of the form "≥" to "≤"
for each constraint $a_i^T x \leq b_i$, add a slack var. s_i
replace constraint by $a_i^T x + s_i = b_i$
& add $s_i \geq 0$

(convince yourself that this works, i.e., if x is a feasible soln. for original LP, $\exists s$ s.t. (x, s) is a feasible soln for modified LP)

Usually, will write LP w/ vector & matrix notation:
$$\min c^T x$$
$$Ax = b$$
$$x \geq 0$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}, \quad A_j \text{ is } j^{th} \text{ column of } A.$$

Some Basic Defns

Given $a \in \mathbb{R}^n$, $a \neq 0$, $b \in \mathbb{R}$
- the set $\{x \in \mathbb{R}^n : a^T x = b\}$ is a hyperplane
- the set $\{x \in \mathbb{R}^n : a^T x \leq b\}$ is a halfspace.

A polyhedron is the intersection of finite half spaces
i.e., given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^n$,
$P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is a polyhedron

(thus, the feasible region of an LP is a polyhedron)

A bounded polyhedron is a polytope.

Basic Feasible Solns.

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron

Defn 1: $x \in P$ is an <u>extreme pt.</u> if it is not a strict convex combination of 2 other pts. in P,
i.e., $\nexists y, z \in P$, $\lambda \in (0,1)$ s.t. $x = \lambda y + (1-\lambda) z$

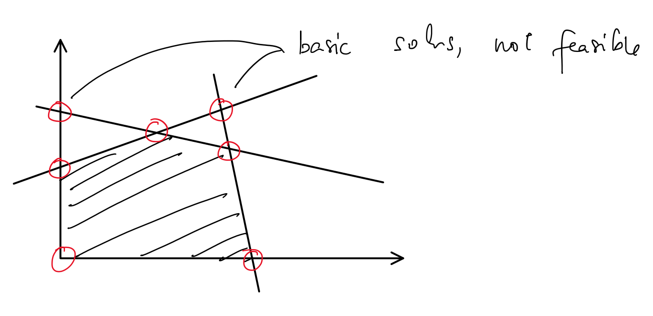
Defn 2: $x \in P$ is a <u>vertex</u> of P if there is a hyperplane that meets P exactly at x,
i.e., $\exists c \in \mathbb{R}^n$ s.t. $c^T x > c^T y \quad \forall y \in P, y \neq x$

Defn 3: (basic feasible soln)
- Constraint $a_i^T x \leq b_i$ is <u>tight</u> at $x^* \in P$ if $a_i^T x^* = b_i$ (or <u>binding</u>, or <u>active</u>)
- Constraints $a_{i_1}^T x \leq b_{i_1}, \ldots, a_{i_k}^T x \leq b_{i_k}$ are linearly independent if $a_{i_1}, \ldots, a_{i_k}$ are l.i.

The $x^* \in P$ is a bfs if n linearly independent constraints are tight at x^*

($x^* \in \mathbb{R}^n$ is a basic soln. if n linearly independent constraints are tight at x^*)

eg.,



Theorem: Let P be a polyhedron, & $x^* \in P$. Then
(1) x^* is an extreme pt $\Leftrightarrow$ (2) x^* is a vertex
$\Leftrightarrow$ x^* is a bfs

Proof: bfs $\Rightarrow$ vertex:
Say constraints $a_1 x = b_1, \ldots, a_n x = b_n$ are l.i., & tight at x^*.
Note that the system of eqns. $a_1 x = b_1, \ldots, a_n x = b_n$ has a unique soln x^*
Hence for all $x \in P$, $x \neq x^*$, $a_i x \leq b_i$, $\forall i \in [n]$
& $a_i x < b_i$ for some $i \in [n]$.
Then let $c = \sum_{i=1}^{n} a_i$. Then $c^T x^* = \sum_i b_i$
$> c^T x \quad \forall x \in P, x \neq x^*$

& hence x^* is a vertex

vertex $\Rightarrow$ extreme pt.
Suppose for a contradiction x^* is a vtx. but not an extreme pt. Then $\exists y, z \in P$ s.t. $x^* = \lambda y + (1-\lambda) z$, $\lambda \in (0,1)$
But since x^* is a vtx., $\exists c \in \mathbb{R}^n$ s.t.
$$c^T x^* > c^T y \quad \& \quad c^T x^* > c^T z$$
$$\Rightarrow c^T x^* > c^T (\lambda y + (1-\lambda) z) = c^T x^*$$

extreme pt $\Rightarrow$ bfs
Again for a contradiction, suppose x^* is an extreme pt. but not a bfs.
(will use: If $a_1, \ldots, a_\ell$ are l.i. vectors, $\ell < n$, $\exists d \in \mathbb{R}^n$, $d \neq 0$ s.t. $\forall i \in [\ell]$, $a_i^T d = 0$)
Let $a_1, \ldots, a_\ell$ be a maximal set of l.i. vectors tight at x^*. Then $\ell < n$. so $\exists d \in \mathbb{R}^n$, $d \neq 0$, s.t. $\forall i \in [\ell]$, $a_i^T d = 0$.
Now consider the pts. $x^* + \varepsilon d$, $x^* - \varepsilon d$ for small ε.
(i) for any constraint $\hat{a} x \leq \hat{b}$ s.t. $\hat{a}$ is a linear combination of $a_1, \ldots, a_\ell$,
$$\hat{a}(x^* \pm \varepsilon d) = \hat{a} x^* \leq \hat{b}$$
(ii) for any constraint $\hat{a} x \leq \hat{b}$ where $\hat{a}$ is l.i. of $a_1, \ldots, a_\ell$, then $\hat{a} x^* < \hat{b}$
& we can choose ε small enough so that
$$\hat{a}(x^* \pm \varepsilon) \leq \hat{b}$$
Hence $x^* \pm \varepsilon \in P$, and x^* is not an extreme pt.

∎

So why are bfs's important?

Theorem: Suppose $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ is nonempty & has an extreme pt. Consider the LP: $\min c^T x : x \in P$
Then either the optimal value is $-\infty$, or there is an optimal extreme pt.

Corollary: If P is a non-empty polytope, there exists a optimal extreme pt.